Asymptotic behavior of entire solutions for degenerate partial differential inequalities on Carnot-Carathéodory metric spaces and Liouville type results

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Abstract. This article is devoted to the study of the behavior of generalized entire solutions for a wide class of quasilinear degenerate inequalities modeled on the following prototype with p-Laplacian in the main part

$$\sum_{m}^{i=1} X_i^*(|\mathbf{X}u|^{p-2}X_iu) \geq |u|^{q-2}u, \quad x \in \mathbb{R}^n,\ q > 1,\ p > 1,$$

where $\mathbb{R}^n$ is a Carnot-Carathéodory metric space, generated by the system of vector fields $\mathbf{X} = (X_1, X_2, .., X_m)$ and X_i^* denotes the adjoint of X_i with respect to Lebesgue measure. For the case where p is less than the homogeneous dimension Q we have obtained a sharp a priori estimate for essential supremum of generalized solutions from below which imply some Liouville-type results.

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1. Introduction

The research subject analyzed in this paper is the asymptotic behavior of nonnegative entire solutions of the following subelliptic differential inequality

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$$\sum_{i=1}^{m} X_i^* A_i(x,u,\mathbf{X}u) \geq f(x,u), \quad x \in R^n, \tag{1.1}$$

under the conditions:

$(\mathbf{A_1})$ $A(x,u,\xi)$ is a Carnot function,
such that for all $\xi \in \mathbb{R}^m : \nu_1|\xi|^p \leq A(x,u,\xi)\xi$ on $\mathbb{R}^n$,

$(\mathbf{A_2})$ $A(x,u,\xi)$ is a Carnot function,
such that for all $\xi \in \mathbb{R}^m : A(x,u,\xi) \leq \nu_2|\xi|^{p-1}$ on $\mathbb{R}^n$,

$(\mathbf{A_3})$ $f(x,u)$ is a Carnot function,
such that for all $u \in \mathbb{R} : f(x,u) \geq \nu_3|u|^{q-2}u$ on $\mathbb{R}^n$,

where $q > 1$, $1 < p < Q$, $\mathbb{R}^n$ is a Carnot-Carathéodory metric space of homogeneous dimension Q, generated by the system of vector fields $\mathbf{X} = (X_1, X_2, .., X_m)$ and X_i^* denotes the adjoint of X_i with respect to Lebesgue measure. $d_{CC}(x)$ is Carnot-Carathéodory metric distance between x and 0. As it was mentioned in the abstract, a typical example of (1.1) is the following

$$\sum_{m}^{i=1} X_i^*(|\mathbf{X}u|^{p-2}X_i u) \geq |u|^{q-2}u, \quad x \in \mathbb{R}^n, \ q > 1, \ 1 < p < Q.$$

Both inequalities are considered under the restriction $u \geq 0$. To describe what kind of results we can expect from such type elliptic problems, it would be interesting to formulate in Euclidean settings some well-known facts. Let us consider the following equation

$$div(|\nabla u|^{p-2}\nabla u) = \nu|u|^{q-2}u, \quad x \in \mathbb{R}^n, \ 1 < q, \ 1 < p < n. \tag{1.2}$$

Here $\mathbb{R}^n$ is Euclidean space and ∇u means a classical notion of gradient of a function u. The simple consideration below illustrate the fact that the asymptotic behavior of entire solutions of this equation tightly depends on exponents p and q.

Let $q < p$. There exists a positive constant ν such that $u(x) = |x|^{p/(p-q)}$ is an entire solution of the equation (1.2) and $\triangle_p u = \nu u^{q-1}$ a.e.(the last fact explains the assumption $q > 1$). This trivial example becomes more interesting in comparison with some Liouville type results obtained by Serrin in [13] for C^1 smooth solutions: if $1 < q < p$ and $u(x) = \bar{\bar{o}}(|x|^{p/(p-q)})$ then $u \equiv 0$ (the description of these results is done according to the notations of this paper and to the case). The example shows that Serrin's condition is sharp and cannot be improved. The closer to p we choose an exponent q in the equation the faster it's solutions must grow at infinity.

In case $q = p$ we could expect that solutions grow faster than any algebraic function and this fact was proved by Serrin in the same paper. Indeed, as examples of solutions, one can consider functions $\exp(x_i)$, $i = 1, 2, .., n$. The method used by Serrin is built on earlier ideas of Redheffer [12] in combination with Serrin's ideas. The approach supposes C^1 continuity of solutions as an important ingredient. A curious reader can find an additional information about some preceding investigations in Euclidean settings also

in papers of Farina [5], Tkachev [14], Brezis [2], Benguria, Lorca and Yarur [1], Haito, Usami [11].

For the case $q > p$ Serrin proved that there could be only trivial solution $u \equiv 0$. This result but for inequalities was treated also by Mitidieri, D'Ambrosio [4] as an auxiliary result under Euclidean settings and by D'Ambrosio [3] on Lie groups.

The method we are going to apply in case $q \leq p$ to prove Liouville type result was introduced in the article of Kondrat'ev and Landis [8] for the case $q > p$ in 1988 and was carefully developed and generalized by Mitidieri and Pohozhaev for the another type of inequalities $-Lu \geq f(x, u)$ in their seminal article [10].

Moreover, the apriori estimate of the maximum of solutions from below is non-trivial. This idea is new even in Euclidean settings. It belongs to Analoti Tedeev, who has obtained the above-mentioned estimate for non-negative solutions of Cauchy problem for parabolic equations with p-Laplacian on Grushin metric spaces (see [9]). Solutions of Cauchy problem for parabolic equations on the space are distributing their mass all over the space with time. The maximum of non-negative solutions is tending to zero. The bound from below in this situation proves that the estimate of maximum is precise.

By the contrary, in the present case we have no decay. We are dealing with entire solutions which are growing close to infinity. The estimate of the maximum from below is much more important than from above.

Let us formulate our main results:

Theorem 1.1. *Let $1 < q < p < Q$ and let u be a weak non-negative entire solution of the differential inequality* (1.1) *then*

i) $\exists\ C_1 > 0$ *and* $R_0 > 0$ *such that for* $\forall\ R \geq R_0$ *on* $d_{CC}-$*annulus* $A_{R/2}^{2R}$

$$||u||_{\infty, A_{R/2}^{2R}} \geq C_1 R^{\frac{p}{p-q}}, \tag{1.3}$$

ii) *if* $u = \bar{o}(d_{CC}^{p/(p-q)}(x))$ *as* $d_{CC}(x) \to +\infty$ *then* $u \equiv 0$.

Remark 1.2. When the a priori estimate (1.3) is proved it is evident that the second assertion of the theorem follows immediately.

Theorem 1.3. *Let $1 < q = p < Q$ and let u be a weak non-negative entire solution of the differential inequality* (1.1) *then*

i) $\exists\ C_2 > 0$ *and* $R_0 > 0$ *such that for* $\forall\ R \geq R_0$

$$||u||_{\infty, A_{R/2}^{2R}} \geq C_2 R^{\frac{p}{p-1}}, \tag{1.4}$$

ii) *Let* $\gamma :\ 0 < \gamma \leq p/(p-1)$ *be an any fixed real number then if* $u = \bar{o}(d_{CC}^{\gamma}(x))$ *as* $d_{CC}(x) \to +\infty$ *then* $u \equiv 0$.

Remark 1.4. The example of Serrin shows us that the estimate (1.4) is not precise. ii) must be true for any positive γ. We are going to prove the assertion *ii)* for the $\gamma \geq p/(p-1)$ in our next paper.

Theorem 1.5. *Let $1 \leq p < q$ and let u be a weak entire solution of the differential inequality* (1.1) *then* $u \equiv 0$.

Remark 1.6. We added this theorem into the paper for the completeness of the picture with the exponents, from one point of view, and because the proof of this result is unexpectedly simple and short using the Kondrat'iev-Landis-Mitidieri-Pohozhaev method, from another point of view.

Remark 1.7. It is also worth to mention that we can formulate Theorem 1.5 not only for p-Laplacian case ($1 \leq p < \infty$) but for all differential operators with non-negative characteristic forms, for example, the theorem is true for mean curvature operator and for total variation operator as well.

The structure of the paper is as follows: in the second section we will present some auxiliary statements and introduce the notion of a generalized entire solution to the equation (1.1). The third section is devoted to the proof of statements *i)* both in Theorem 1.1 and in Theorem 1.3. In the fourth section, we prove Theorem 1.5.

2. Auxiliary results

We consider in $\mathbb{R}^n$ a system $\mathbf{X} = (X_1, X_2, .., X_m)$ of vector fields

$$X_j = \sum_{k=1}^{n} b_{jk}(x) \frac{\partial}{\partial x_k}, \quad j = 1, .., m,$$

having real-valued, locally Lipschitz-continuous b_{jk}. Through the paper if u is a non smooth then $X_j u$ will be meant in the distributional sense. For the system $\mathbf{X} = (X_1, X_2, .., X_m)$ we denote by $|\mathbf{X}u| = \left(\sum\limits_{j=1}^{m} (X_j u)^2 \right)^{1/2}$ the length of the horizontal gradient

$$\mathbf{X}u = (X_1 u, X_2 u, .., X_m u).$$

Thus let us consider also $\mathbb{R}^n$ as a Carnot-Carathéodory metric space of a homogeneous dimension Q with Carnot-Carathéodory metric distance $d_{CC}(x)$ generated by the system of vector fields $\mathbf{X} = (X_1, X_2, .., X_m)$ as it was defined in [6]. Let us define a $d_{CC}-$annulus $A_r^R := \{x \in \mathbb{R}^n : \; r < d_{CC}(x) < R\}$.

Let us introduce $\mathfrak{L}_{1,p}(\Omega)$ for $1 \leq p < \infty$ as follows: it is a weak Sobolev space, which is a norm closure of a set of functions $C^\infty(\bar{\Omega})$ with the norm

$$\|f\|_{\mathfrak{L}^{1,p}(\Omega)} = \left(\int\limits_{\Omega} \left(|\mathbf{X}f|^p + |f|^p \right) dx \right)^{\frac{1}{p}},$$

which is an equivalent to the norm

$$\left(\int\limits_{\Omega} |\mathbf{X}f|^p dx \right)^{\frac{1}{p}} + \left(\int\limits_{\Omega} |f|^r dx \right)^{\frac{1}{r}},$$

where $0 < r \leq p$.

$\overset{\circ}{\mathfrak{L}}_{1,p}(\Omega)$ is a subspace of $\mathfrak{L}_{1,p}(\Omega)$, which is a norm closure of functions from $C_0^\infty(\Omega)$ with the norm of $\mathfrak{L}_{1,p}(\Omega)$.

It is well known that for $\mathfrak{L}_{1,p}(\Omega)$ and $\overset{\circ}{\mathfrak{L}}_{1,p}(\Omega)$ under Carnot-Carathéodory spaces were proved embedding Sobolev type theorems and Nirenberg-Galiardo type inequalities. We would refer the reader interested in the theory of Carnot-Carathéodory spaces to the survey [6], where one can find more useful references. Particularly, from the results of the survey [6] one can easy prove the following multiplicative Nirenberg-Galiardo type inequality.

Proposition 2.1. *For every function $f \in \mathfrak{L}_{1,p}(\mathbb{R}^{N+M})$, the following inequality holds*

$$\int_{\mathbb{R}^{N+M}} |f|^p dz \leq C_{14} \left(\int_{\mathbb{R}^{N+M}} |\mathbf{X} f|^p dz \right)^{\beta_1} \left(\int_{\mathbb{R}^{N+M}} |f|^{\beta_2} dz \right)^{\frac{(1-\beta_1)p}{\beta_2}}. \tag{2.1}$$

Here $0 < \beta_1 = \beta_1(\beta_2) < 1$.

Definition 2.2. We can say that a function $u(x)$ as a generalized entire solution (a weak solution) of the equation (1.1) if $u(x) \in \mathfrak{L}_{1,p,loc}(\mathbb{R}^n)$ and satisfies the inequality

$$\int_\Omega \sum_{i=1}^m A_i(x, u, \mathbf{X}u) X_i \varphi + f(x, u) \varphi dx \leq 0, \tag{2.2}$$

for any $\varphi \in \overset{\circ}{\mathfrak{L}}_{1,p}(\Omega)$ and Ω is an open bounded domain from $\mathbb{R}^n$.

3. Proof of a priori estimates for the case $q \leq p$

In this section we will prove results indicated by i) in Theorem 1.1 and Theorem 1.3. At first let us mention that using the simple change of notation for u it is easy to check that conditions $(\mathbf{A_1})$, $(\mathbf{A_2})$, $(\mathbf{A_3})$ could be rewritten in the following form

$(\mathbf{A_1'})$ $A(x, u - \tilde{b}, \xi)$ is a Carnot function, such that for all $\xi \in \mathbb{R}^m : \nu_1 |\xi|^p \leq A(x, u - \tilde{b}, \xi)\xi$ on $\mathbb{R}^n$,

$(\mathbf{A_2'})$ $A(x, u - \tilde{b}, \vec{\xi})$ is a Carnot function, such that for all $\xi \in \mathbb{R}^m : A(x, u - \tilde{b}, \xi) \leq \nu_2 |\xi|^{p-1}$ on $\mathbb{R}^n$,

$(\mathbf{A_3'})$ $f(x, u)$ is a Carnot function, such that for all $u \in \mathbb{R} : f(x, u - \tilde{b}) \geq \nu_3 |u - \tilde{b}|^{q-2}(u - \tilde{b})$ on $\mathbb{R}^n$

where $\tilde{b}$ is a fixed positive constant. Let us define a cutoff function ξ_R from the space of d_{CC}-Lipschitz continuous functions such that $\xi_R \equiv 0$ if $d_{CC}(x) \leq R$ or $d_{CC}(x) \geq \mu R$ with $|\mathbf{X}\xi_R| \leq C/(\mu - 1)R$ a.e. in $\mathbb{R}^n$ for any $R > 0, \mu > 1$. This function exists according to [7].

Then we can use $(u - \tilde{b})_+^s \xi_R^m$ ($\tilde{b} > 0,\ m, s > 1$ we shall choose in what follows) as a test function for integral inequality (2.2) which together with standard Young's inequality yields basic for the case $p \neq q$ a priori estimate

Lemma 3.1. *Let u be a weak solution of the inequality* (1.1). *Then the following a priori estimate holds locally on any annulus $A_R^{\mu R}$, where $R>0$ is big enough and $\mu>1$ and for any level number $\tilde b>0$ and exponents $m,s>1$*

$$\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^{q+s-1}\xi_R^m dx + \int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} |\mathbf{X}(u-\tilde b)_+^{\frac{p+s-1}{p}}|^p\xi_R^m dx$$

$$\leq C\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^{p+s-1}|\mathbf{X}\xi_R|^p\xi_R^{m-p}dx, \tag{3.1}$$

where $C=C(\varepsilon,m,p,s)$, $m>p$ and $\varepsilon>0$ is small enough.

To investigate the case $p=q$ we will need the following

Lemma 3.2. *Let u be a weak solution of the inequality* (1.1) *for $1<q=p<Q$. Then the following a priori estimate holds locally on any annulus $A_R^{\mu R}$, where $R>0$ is big enough and $\mu>1$ and for any level number $\tilde b>0$*

$$\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^{p-1}\xi_R^m dx + \int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} |\mathbf{X}(u-\tilde b)_+^{\frac{2(p-1)}{p}}|^p\xi_R^m dx$$

$$\leq C\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^{2(p-1)}|\mathbf{X}\xi_R|^p\xi_R^{m-p}dx, \tag{3.2}$$

where $C=C(\varepsilon_1,m,p)$, $m>p$ and $\varepsilon_1>0$ is small enough.

Indeed, let us take $\frac{(u-\tilde b)_+}{(u-\tilde b)_++\bar\varepsilon}\xi_R^m$ ($\bar\varepsilon>0$) as a test function, then using the same argument as in the previous lemma we obtain

$$\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} \frac{(u-\tilde b)_+^p}{(u-\tilde b)_++\bar\varepsilon}\xi_R^m dx + (\bar\varepsilon-\varepsilon_1)\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} \frac{|\mathbf{X}(u-\tilde b)_+|^p}{\Big((u-\tilde b)_++\bar\varepsilon\Big)^2}\xi_R^m dx$$

$$\leq C(\varepsilon_1)m^p\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^p((u-\tilde b)_++\bar\varepsilon)^{p-2}|\mathbf{X}\xi_R|^p\xi_R^{m-p}dx. \tag{3.3}$$

Throwing away the second term from the left and letting $\bar\varepsilon\to 0$ we obtain

$$\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^{p-1}\xi_R^m dx \leq C(\varepsilon_1)m^p\int\limits_{A_R^{\mu R}\cap\{u\geq\tilde b\}} (u-\tilde b)_+^{2(p-1)}|\mathbf{X}\xi_R|^p\xi_R^{m-p}dx. \tag{3.4}$$

Choosing $s=p-1$ in (3.1) and taking into account that $q=p$ we can compose the required inequality from (3.1) and (3.4).

The assumptions of the next lemma are satisfied both for the case $p\neq q$ and $p=q$.

Lemma 3.3. *Suppose that for all weak solutions of the inequality* (1.1) *for some fixed* $l > 0$ *and* $0 < \delta < 1$ *the following a priori estimate holds locally on any annulus* $A_{\tilde{R}}^{\mu\tilde{R}}$, *where* $\tilde{R} > 0$ *is big enough and* $\mu > 1$ *and for any level number* $\tilde{b} > 0$

$$\int\limits_{A_{\tilde{R}}^{\mu\tilde{R}}\cap\{u\geq\tilde{b}\}} (u-\tilde{b})_+^l \xi_{\tilde{R}}^m dx + \int\limits_{A_{\tilde{R}}^{\mu\tilde{R}}\cap\{u\geq\tilde{b}\}} |\mathbf{X}(u-\tilde{b})_+^{\frac{l(1+\delta)}{p}}|^p \xi_{\tilde{R}}^m dx$$

$$\leq C \int\limits_{A_{\tilde{R}}^{\mu\tilde{R}}\cap\{u\geq\tilde{b}\}} (u-\tilde{b})_+^{l(1+\delta)} |\mathbf{X}\xi_{\tilde{R}}|^p \xi_{\tilde{R}}^{m-p} dx. \tag{3.5}$$

Here p *is the parameter of the inequality* (1.1) *and* $p < Q$, m *is any fixed number greater than* $2p$, $C = C(m,p,\delta,Q)$ *is some constant usually larger than* 1. *Then for any* $R > 0$, *for any fixed* $\sigma_1,\ \sigma_2,\ \sigma_3 > 0$ *and for any level numbers* $b_1 > \tilde{b} > b_2 > 0$ *and* $0 < \nu < 1$ *one can find proper* $\beta_1 \in (0,1)$ *which depends only from* p, l, δ *and* ν *such that*

$$\int\limits_{A_{R+\sigma_1 R}^{R+\sigma_1 R+\sigma_2 R}} (u-b_1)_+^l \xi_R^m dx \leq C(\sigma_{min}R)^{-\frac{p}{1-\beta_1}} \left(\int\limits_{A_R^{R+\sigma_1 R+\sigma_2 R+\sigma_3 R}} (u-b_2)_+^{l\nu} \xi_R^m dx \right)^{\frac{1+\delta}{\nu}} \tag{3.6}$$

Suppose that $r_j^l = R(1+\sigma_1 2^{-j})$, $r_j^r = R(1+\sigma_1+\sigma_2+\sigma_3-\sigma_3 2^{-j})$, $b_j = b_2 + (b_1-b_2)2^{-j}$, $\tilde{b}_{j+1} = (b_j + b_{j+1})/2$, $j = 0,1,2,..$ and let $\xi_{j+1}(x)$ be a d_{CC}- Lipschitz continuous function such that $\xi_{j+1}(x) \in \overset{\circ}{\mathfrak{L}}_{1,p}(\mathbb{R}^n)$, $\xi_{j+1} \equiv 1$ for all $x \in A_{r_j^l}^{r_j^r}$, $\xi_{j+1} \equiv 0$ for all x if $r_{j+1}^r \leq d_{CC}(x)$ or when $d_{CC}(x) \leq r_{j+1}^l$, $|\mathbf{X}\xi_{j+1}| = 2^j|\mathbf{X}\xi_j|$, $|\mathbf{X}\xi_{j+1}| \leq C/(r_j^l - r_{j+1}^l)$ for a. e. $x \in \mathbb{R}^n$. This function exists according to [7]. Setting in the inequality (3.5) $\tilde{R} = r_{j+1}^l, \mu\tilde{R} = r_{j+1}^r, \tilde{b} = \tilde{b}_{j+1}, \xi_{\tilde{R}} = \xi_{j+1}, A_{r_{j+1}^l}^{r_{j+1}^r} = A_{j+1}$, we obtain

$$\int\limits_{A_{j+1}\cap\{u\geq\tilde{b}_{j+1}\}} (u-\tilde{b}_{j+1})_+^l \xi_{j+1}^m dx + \int\limits_{A_{j+1}\cap\{u\geq\tilde{b}_{j+1}} |\mathbf{X}(u-\tilde{b}_{j+1})_+^{\frac{l(1+\delta)}{p}}|^p \xi_{j+1}^m dx$$

$$\leq C \int\limits_{A_{j+1}\cap\{u\geq\tilde{b}_{j+1}\}} (u-\tilde{b}_{j+1})_+^{l(1+\delta)} |\mathbf{X}\xi_{j+1}|^p \xi_{j+1}^{m-p} dx. \tag{3.7}$$

Denote $f_j = (u-b_j)_+^{\frac{l(1+\delta)}{p}} \xi_j^{\frac{m-p}{p}}$. Then using this and previous notations and listed above properties of cut-off functions ξ_j and ξ_{j+1}, we can get

$$\int\limits_{A_j} |f_j|^{\frac{p}{1+\delta}} dx + \int\limits_{A_j} |\mathbf{X}f_j|^p dx \leq \int\limits_{A_{j+1}\cap\{u\geq\tilde{b}_{j+1}\}} (u-\tilde{b}_{j+1})_+^l \xi_{j+1}^m dx$$

$$+\int\limits_{A_{j+1}\cap\{u\geq\tilde{b}_{j+1}}|\mathbf{X}(u-\tilde{b}_{j+1})_+^{\frac{l(1+\delta)}{p}}|^p\xi_{j+1}^m dx+ \tag{3.8}$$
$$\left(\frac{m-p}{p}\right)^p\int\limits_{A_j}(u-\tilde{b}_j)_+^{l(1+\delta)}|\mathbf{X}\xi_{j+1}|^p\xi_{j+1}^{m-p}dx.$$

Then it follows from (3.7) and (3.8) that

$$\int\limits_{A_j}|f_j|^{\frac{p}{1+\delta}}dx+\int\limits_{A_j}|\mathbf{X}f_j|^p dx\leq C\frac{2^{pj}}{(\sigma_{min}R)^p}\int\limits_{A_{j+1}}|f_{j+1}|^p dx \tag{3.9}$$

To estimate from above the integral on the right hand side of the inequality (3.9) we need to use the Nirenberg-Gagliardo type multiplicative inequality (2.1). Let us apply it to the function $f=f_{j+1}$ with $\beta_2=p/(1+\delta)$. This yields

$$\int\limits_{A_j}|f_j|^{\frac{p}{1+\delta}}dx+\int\limits_{A_j}|\mathbf{X}f_j|^p dx\leq \tag{3.10}$$
$$C\frac{2^{pj}}{(\sigma_{min}R)^p}\left(\int\limits_{A_{j+1}}|\mathbf{X}f_{j+1}|^p dx\right)^{\beta_1}\left(\int\limits_{A_{j+1}}|f_{j+1}|^{\frac{p\nu}{1+\delta}}dx\right)^{\frac{(1-\beta_1)(1+\delta)}{\nu}},$$

where, after calculations, $\beta_1=\frac{\frac{1+\delta}{\nu}-1}{\frac{1+\delta}{\nu}-1+\frac{p}{Q}}<1$.

Using the standard Young's inequality with exponents β_1^{-1} and $(1-\beta_1)^{-1}$ together with (3.9), we have

$$\int\limits_{A_j}|f_j|^{\frac{p}{1+\delta}}dx+\int\limits_{A_j}|\mathbf{X}f_j|^p dx\leq\varepsilon_2\int\limits_{A_{j+1}}|\mathbf{X}f_{j+1}|^p dx+$$
$$\frac{C(\varepsilon_2)\left(2^{\frac{p}{1-\beta_1}}\right)^j}{(\sigma_{min}R)^{\frac{p}{1-\beta_1}}}\left(\int\limits_{A_{j+1}}|f_{j+1}|^{\frac{p\nu}{1+\delta}}dx\right)^{\frac{1+\delta}{\nu}},$$

where $\varepsilon_2>0$ is a small constant to be chosen later. By induction we obtain

$$\int\limits_{A_0}|f_0|^{\frac{p}{1+\delta}}dx+\int\limits_{A_0}|\mathbf{X}f_0|^p dx\leq\varepsilon_2^{j+1}\int\limits_{A_{j+1}}|\mathbf{X}f_{j+1}|^p dx+$$
$$\frac{C(\varepsilon_2)\sum_{k=0}^{j}\left(\varepsilon_2 2^{\frac{p}{1-\beta_1}}\right)^k}{(\sigma_{min}R)^{\frac{p}{1-\beta_1}}}\left(\int\limits_{A_{j+1}}|f_{j+1}|^{\frac{p\nu}{1+\delta}}dx\right)^{\frac{1+\delta}{\nu}}.$$

Here we can cast aside the gradient term from the left. Then according to the definition of weak solutions the integral $\int_{A_{j+1}}|\mathbf{X}f_{j+1}|^p dx$ is convergent. Choosing ε_2 so that $\varepsilon_2 2^{\frac{p}{1-\beta_1}}=1/2<1$ and letting $j\to\infty$, we obtain the required estimate.

Lemma 3.4. *Under the assumptions of the previous lemma the following inequality holds*

$$\|u\|_{\infty,\ A^{\bar R+\bar\sigma_1\bar R+\bar\sigma_2\bar R}_{\bar R+\bar\sigma_1\bar R}} \le C(\bar\sigma_{min}\bar R)^{-\frac{p}{1-\beta_1}\cdot\frac{1}{l(1-\nu)}}\left(\int\limits_{A^{\bar R+\bar\sigma_1\bar R+\bar\sigma_2\bar R+\bar\sigma_3\bar R}_{\bar R}}|u|^{l\nu}dx\right)^{\frac{1}{l\nu}\cdot\frac{1+\delta-\nu}{1-\nu}}. \tag{3.11}$$

Suppose that $R^l_i=\bar R(1+\bar\sigma_1-\bar\sigma_1 2^{-i})$, $\tilde R^l_i=(R^l_i+R^l_{i+1})/2$, $R^r_i=\bar R(1+\bar\sigma_1+\bar\sigma_2+\bar\sigma_3 2^{-i})$, $\tilde R^r_i=(R^r_i+R^r_{i+1})/2$, $\bar\sigma_{min}=\min(\bar\sigma_1;\bar\sigma_3)$, $h_i=k(1-2^{-i-1})$, $\tilde h_i=(h_i+h_{i+1})/2$ for $i=0,1,2..$ Then under the settings $R=R^l_i$, $R+\sigma_1R=\tilde R^l_i$, $R+\sigma_1R+\sigma_2R=\tilde R^r_i$, $R+\sigma_1R+\sigma_2R+\sigma_3R=R^r_i$, $b_1=\tilde h_i$, $b_2=h_i$, $A^{R+\sigma_1R+\sigma_2R}_{R+\sigma_1R}=A^{\tilde R^r_i}_{\tilde R^l_i}=\tilde A_i, A^{R+\sigma_1R+\sigma_2R+\sigma_3R}_{R}=A^{R^r_i}_{R^l_i}=A_i$ Lemma 3.3 implies

$$\int\limits_{\tilde A_i\cap\{u\ge\tilde h_i\}}(u-\tilde h_i)^l_+dx\le C(\tilde R^l_i-R^l_i)^{-\frac{p}{1-\beta_1}}\left(\int\limits_{A_i\cap\{u\ge\tilde h_i\}}(u-h_i)^{l\nu}_+dx\right)^{\frac{1+\delta}{\nu}} \tag{3.12}$$

Let us denote $I_{i+1}=\int\limits_{A_{i+1}}(u-h_{i+1})^{l\nu}_+dx$. Thus, from (3.12) one can get the following estimate

$$I_{i+1}\le C(\tilde R^l_i-R^l_i)^{-\frac{p}{1-\beta_1}}(h_{i+1}-h_i)^{-l(1-\nu)}I_i^{\frac{1+\delta}{\nu}}.$$

It means that the sequence $\{I_i\}$ satisfies the assumptions of Ladizhenskaya's Lemma and from the last inequality we have $I_{i+1}\le C_{Lad}b^iI_i^{1+\theta}$, denoting $\theta=\frac{1+\delta}{\nu}-1>0$,

$$C_{Lad}=C(\bar\sigma_{min}\bar R)^{-\frac{p}{1-\beta_1}}(2^{\frac{p}{1-\beta_1}+l(1-\nu)})^ik^{-l(1-\nu)}>0,\ b=2^{\frac{p}{1-\beta_1}+l(1-\nu)}>1,$$

where the constant C_{Lad} is controlled by k. Then Ladizhenskay's Lemma implies that $I_i\to0$ as $i\to\infty$, and $\|u\|_{\infty,A_\infty}\le k$ if $I_0\le C_{Lad}^{-1/\theta}b^{-1/\theta^2}$. Choosing

$$k=C(\bar\sigma_{min}\bar R)^{-\frac{p}{1-\beta_1}\cdot\frac{1}{l(1-\nu)}}\left(\int\limits_{A^{\bar R+\bar\sigma_1\bar R+\bar\sigma_2\bar R+\bar\sigma_3\bar R}_{\bar R}}|u|^{l\nu}dx\right)^{\frac{1}{l\nu}\cdot\frac{1+\delta-\nu}{1-\nu}},$$

where constant C is sufficiently large, we obtain the required estimate. The lemma 3.4 is proved.

To make a simple iteration process, redenoting manifest that together with Lemma 3.4, we have the following inequality on any sequence of nested

annuli A_i such that $A_{\mu R}^{2\mu R} = A_\infty \subset A_{i+1} \subset A_i \subset A_0 = A_R^{3\mu R}$

$$\|u\|_{\infty,A_{i+1}} \le Cb^i R^{-\frac{p}{1-\beta_1}\cdot\frac{1}{l(1-\nu)}} \left(\int_{A_i} |u|^{l\nu} dx\right)^{\frac{1}{l\nu}\cdot\frac{1+\delta-\nu}{1-\nu}} \le \tag{3.13}$$

$$Cb^i R^{-\frac{p}{1-\beta_1}\cdot\frac{1}{l(1-\nu)}} R^{\frac{n(\frac{1+\delta}{\nu}-1)}{l(1-\nu)}} \|u\|_{\infty,A_i}^{1+\frac{\delta}{1-\nu}}.$$

Let us denote also $c = CR^{-\frac{p}{1-\beta_1}\cdot\frac{1}{l(1-\nu)}} R^{\frac{n(\frac{1+\delta}{\nu}-1)}{l(1-\nu)}}$, $\varepsilon = \frac{\delta}{1-\nu}$. Thus,

$$\|u\|_{\infty,A_{i+1}} \le cb^i \|u\|_{\infty,A_i}^{1+\varepsilon},$$

$$\|u\|_{\infty,A_\infty} \le \|u\|_{\infty,A_i} \le c^{\frac{(1+\varepsilon)^i-1}{\varepsilon}} b^{\frac{(1+\varepsilon)^i-1}{\varepsilon^2}-\frac{i}{\varepsilon}} \|u\|_{\infty,A_0}^{(1+\varepsilon)^i},$$

$$\|u\|_{\infty,A_{\mu R}^{2\mu R}}^{\frac{1}{(1+\varepsilon)^i}} \le c^{\frac{1}{\varepsilon}-\frac{1}{\varepsilon(1+\varepsilon)^i}} b^{\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon^2(1+\varepsilon)^i}-\frac{i}{\varepsilon(1+\varepsilon)^i}} \|u\|_{\infty,A_R^{3\mu R}}.$$

As far as $\|u\|_{\infty,A_{\mu R}^{2\mu R}} \neq 0$ is a fixed number for any fixed R, we obtain as $i \to \infty$

$$1 \le c^{\frac{1}{\varepsilon}} b^{\frac{1}{\varepsilon^2}} \|u\|_{\infty,A_R^{3\mu R}},$$

$$1 \le cb^{\frac{1}{\varepsilon}} \|u\|_{\infty,A_R^{3\mu R}}^{\varepsilon}.$$

This yields

$$R^{\frac{p}{1-\beta_1}\cdot\frac{1}{l(1-\nu)}-\frac{n(\frac{1+\delta}{\nu}-1)}{l(1-\nu)}} \le C(\mu,n)\|u\|_{\infty,A_R^{3\mu R}}^{\frac{\delta}{1-\nu}}, \tag{3.14}$$

and after the appropriate calculations, from (3.14) we have the desired result

$$R^{\frac{p}{l\delta}} \le C\|u\|_{\infty,A_R^{3\mu R}}.$$

Now we need only to choose appropriate parameters such as l and δ according to $q < p$ case and for the case $q = p$.

4. Proof of Theorem 1.5

Here we will get the a priori estimate with $p < q$. Let u be a weak solution of the inequality (1.1). Following the same way as it was in the proof of Lemma 3.1 and using non-negativity of characteristic form of our differential operator, we can obtain

$$\int_\Omega |u|^{q+s-1} \xi_R^m dx \le \frac{m^p}{s2^p} \int_\Omega |u|^{p+s-1} |\mathbf{X}\xi_R|^p \xi_R^{m-p} dx.$$

It gives us together with Holder inequality the following estimate

$$\int_\Omega |u|^{q+s-1} \xi_R^m dx \le \frac{m^p}{s2^p(R-r)^p} C(Q) R^{\frac{Q(q-p)}{q+s-1}} \left(\int_\Omega |u|^{q+s-1} \xi_R^{\frac{(m-p)(q+s-1)}{p+s-1}} dx\right)^{\frac{p+s-1}{q+s-1}}.$$

Let us fix $m = \frac{p(q+s-1)}{q-p}$ such that it satisfies $m = \frac{(m-p)(q+s-1)}{p+s-1}$ then integrals from both sides are equal. Hence, we have

$$\left(\int_{B_R} |u|^{q+s-1}\xi_R^{\frac{p(q+s-1)}{q-p}}\,dx\right)^{\frac{q-p}{q+s-1}} \le \frac{m^p C(Q) R^{\frac{Q(q-p)}{q+s-1}}}{s2^p(R-r)^p} =$$

$$C\frac{R^{\frac{Q(q-p)}{q+s-1}}}{(R-r)^p}\frac{p^p}{(q-p)^p}(1+\frac{q-1}{s})(q+s-1)^{p-1}.$$

This yields

$$\left(\int_{B_R} |u|^{q+s-1}\xi_R^{\frac{p(q+s-1)}{q-p}}\,dx\right)^{\frac{q-p}{q+s-1}} \le \frac{C(p,q)R^{\frac{Q(q-p)}{q+s-1}}(q+s-1)^{p-1}}{(R-r)^p}(1+\frac{q-1}{s}).$$

Let us choose $r = R/2$ and redenote $\bar{s} = q+s-1$. Then we have

$$\left(\int_{B_{R/2}} |u|^{\bar{s}}dx\right)^{\frac{1}{\bar{s}}} \le C(p,q)\left(\frac{R^{\frac{Q(q-p)}{\bar{s}}}\bar{s}^{p-1}}{R^p}\right)^{\frac{1}{q-p}}(1+\frac{q-1}{\bar{s}-q+1})^{\frac{1}{q-p}}.$$

Now, as $R \to +\infty$ then for all $\bar{s} > \frac{Q(q-p)}{p}$ we have $L_{\bar{s}}$-norms on $\mathbb{R}^n$ are zero. Thus, Theorem 1.5 is proved.

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